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NY—391—2023

FACULTY OF ARTS/SCIENCE

M.A./M.Sc. (Second Year) (Fourth Semester) EXAMINATION

NOVEMBER/DECEMBER, 2023

(New/CBCS Pattern)

MATHEMATICS

Paper XXIII(A)

(Integral Equations)

(Friday, 15-12-2023) **Time : 2.00 p.m. to 5.00 p.m.**

Time—Three Hours *Maximum Marks*—75

Note :— (*i*) *All* questions are compulsory.

(*ii*) Figures to the right indicate full marks.

1. Attempt the following : 15

(*a*) Consider the differential equation :

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n(x)\,y = \phi(x)$$

with the initial conditions

$$y(a) = q_0,\ y'(a) = q_1,\ y''(a) = q_2,\ \ldots,\ y^{(n-1)}(a) = q_{n-1}$$

Obtain an equivalent Volterra integral equation, where $a_1(x)$, $a_2(x)$, ..., $a_n(x)$ and $\phi(x)$ are defined and continuous in $a \le x \le b$.

P.T.O.

(*b*) Reduce $y''(x) - \lambda y(x) = 0$ with initial conditions $y(0) = 0$, $y(l) = 0$ to a Fredholm integral equation. Conversely, recover the original differential equation with the initial conditions from the integral equation obtained.

2. Attempt the following : 15

(*a*) Solve :

$$y(x) = f(x) + \lambda \int_{-1}^{1} (xt + x^2t^2)\, y(t)\, dt.$$

Or

(*b*) Explain the method of solving the Fredholm integral equation :

$$y(x) = f(x) + \lambda \int_{a}^{b} k(x,t)\, y(t)\, dt$$

where the kernel $k(x, t)$ is separable. Hence solve

$$y(x) = e^x + \lambda \int_{0}^{1} 2e^x e^t y(t)\, dt.$$

3. Attempt the following : 15

(*a*) Prove that the set of eigenvalues of the second iterated kernel coincide with the set of squares of the eigenvalues of the given kernel.

Or

(*b*) Prove that the eigenfunctions of a symmetric kernel of a Fredholm integral equation of the first kind, corresponding to different eigenvalues are orthogonal.

4. Attempt the following : 15

(*a*) Solve the Volterra integral equation of the first and second kind with convolution type kernel using Laplace transform. Prove that if the kernel of the Volterra integral equation is of convolution type, then its resolvent kernel is also of convolution type. Using Laplace transform, obtain the expression for the resolvent kernel of Volterra integral equation of second kind with convolution type kernel.

Or

(*b*) Explain the method of solving the Abel singular integral equation

$$f(x) = \int_0^x \frac{y(t)}{(x-t)^\alpha} dt, \qquad 0<\alpha<1$$

in which $f(x)$ is a known function while $y(t)$ is to be determined. Hence solve the equation

$$x = \int_0^x \frac{y(t)}{(x-t)^{1/2}} dt.$$

5. Attempt any *three* of the following : 15

(*a*) Convert the initial value problem into an integral equation :

$$y'' + x\,y = 1,\ y\,(0) = 0,\ y'\,(0) = 0.$$

(*b*) Solve the integral equation :

$$y\,(x) = \tan x + \int_{-1}^{1} e^{\sin^{-1} x}\, y(t)\,dt\,.$$

(*c*) Prove that the sequence of eigenfunctions of a symmetric kernel can be made orthonormal.

(*d*) Using Laplace transform, find the resolvent kernel of the integral equation :

$$\mathrm{Y}\,(t) = \mathrm{F}(t) + \int_{0}^{t} e^{t-x}\,\mathrm{Y}(x)\,dx\,.$$