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NY—136—2023

FACULTY OF ARTS AND SCIENCE

M.A./M.Sc. (Second Year) (Fourth Semester)

(NewCBCS Pattern)

MATHEMATICS

(Paper–XX)

(Abstract Algebra-II) (Field Theory)

(Friday, 08-12-2023) **Time : 2.00 p.m. to 5.00 p.m.**

Time—3 *Hours* *Maximum Marks*—75

N.B. :— (*i*) *All* questions are compulsory.

(*ii*) Figures to the right indicate full marks.

1. Attempt the following : 15

(*a*) If $F \subseteq E \subseteq K$ be field such that $[K : E] < \infty$ and $[E : F] < \infty$, then prove the following :

(*i*) $[K : F] < \infty$

(*ii*) $[K : F] = [K : F] [E : F]$.

(*b*) Show that, the polynomial $f(x) = x^4 - 2$ is irreducible over Q.

Or

(*a*) State and prove Gauss lemma.

(*b*) Check the following polynomial are irreducible over Q or not :

(*i*) $f(x) = x^3 - 5x + 10$

(*ii*) $g(x) = x^2 + x + 1$.

P.T.O.

2. Attempt the following : 15

(*a*) Prove that, any finite field F with P^n elements is the splitting field of $x^{p^n} - x \in F_p(x)$,

consequently any two finite field with P_n elements are isomorphic.

(*b*) Prove that $Q\left(\sqrt{2}, \sqrt{3}\right) = Q\left(\sqrt{2} + \sqrt{3}\right)$.

Or

(*a*) Let $f(x) \in F[x]$ be any polynomial of degree $n \geq 1$ with α is root, then prove that α is multiple root of $f(x)$ if and only if $f'(d) = 0$.

(*b*) Find the splitting field of polynomial $f(x) = x^2 - 2$ over Q.

3. Attempt the following : 15

(*a*) If E is finite extension of a field F, then prove that :

$|G(E/F)| \leq |E : F|$.

(*b*) Let F be a field of characteristic $\neq 2$. Let $x^2 - a \in F[x]$ be an irreducible polynomial over F, then show that its Galois group is of order 2.

Or

(*a*) Let E be the Galois extension of F. Let K be any subfield of E containing F. Then the mapping $K \to G(E/K)$ setup a one-one corresponding from set of subfield of E containing F to the subgroup of G (E/F). Then prove that :

(*i*) $K = E_{G(E/K)}$

(*ii*) For any subgroup H of G(E/F), then $H = G(E/E_H)$.

(*b*) Prove that, the group $G(Q(\alpha)/Q)$, where $x^5 = 1$ and $\alpha \neq 1$ is isomorphic to cyclic group of order 4.

4. Attempt the following : 15

(*a*) Let F be a field and n be a positive integer, then prove that primitive nth root of unity is some extension E of F if and only either char F = 0 or char $F \nmid n$.

(*b*) If a and b are constructible number, then prove that :

(*i*) $a.\ b$ is constructible

(*ii*) a/b *is* constructible if $a/b \neq 0$.

Or

(*a*) If $a > 0$ is constructible, then prove that $\sqrt{a}$ is constructible.

(*b*) Express the following symmetric polynomial has a rational function of elementary symmetric function :

$$x_1^2 + x_2^2 + x_3^2.$$

5. Attempt any *three* of the following : 15

(*a*) Determine the minimal polynomial over Q of the element $\sqrt{2} + 5$.

(*b*) Show that, R is not normal extension of Q.

(*c*) Let E_H be a subfield of F and if E is an extension of a field F and $H \leq G$, then show that $F \subseteq E_H \subseteq H$.

(*d*) Prove that, the Galois group of $x^4 + x^2 + 1$ is the same as that of $x^6 - 1$ and is of order 2.