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NY—18—2023

FACULTY OF ARTS AND SCIENCE

M.A./M.Sc. (Second Year) (Third Semester) EXAMINATION

NOVEMBER/DECEMBER, 2023

(New/CBCS Pattern)

MATHEMATICS

Paper–XIII

(Functional Analysis)

(Tuesday, 5-12-2023) **Time : 2.00 p.m. to 5.00 p.m.**

Time—3 *Hours* *Maximum Marks*—75

N.B. :— (*i*) *All* questions are compulsory.

(*ii*) Figures to the right indicate full marks.

1. Attempt the following :

(*a*) (*i*) State and prove open mapping theorem. 8

(*ii*) State and prove Hahn-Banach theorem. 7

Or

(*b*) If M is closed linear subspace of normed linear space N, then prove that $\frac{N}{M} = \{x + M \mid x \in N\}$ is a normed linear space with respect to the norm defined as : 15

$$\|x + M\|' = \inf \{\|x + m\| : n \in M\}$$

Moreover prove that, $\frac{N}{M}$ is a Banach space if N is a Banach space.

P.T.O.

2. Attempt the following :

(*a*) (*i*) Define orthonormal set. State and prove Bessel's inequality for finite orthonormal set. 8

(*ii*) State and prove parallelogram law. 7

Or

(*b*) (*i*) State and prove Riesz-Representation theorem. 8

(*ii*) Show that the mapping $\psi : H \to H^*$ defined by $\psi(y) = f_y \quad \forall\, y \in H$, where $f_y \in H^*$ is defined as $f_y(x) = \langle x, y \rangle \quad \forall x \in H$ is : 7

(*i*) one-one

(*ii*) additive

(*iii*) non-linear

(*iv*) onto

(*v*) isometry.

3. Attempt the following :

(*a*) Define adjoint of an operator. If $T^* \in B(H)$, then prove that : 15

(*i*) $(T_1 + T_2)^* = T_1{}^* + T_2{}^*$

(*ii*) $(\alpha T)^* = \bar{\alpha} T^*$

(*iii*) $(T_1T_2)^* = T_2^*T_1^*$

(*iv*) $(T^*)^* = T^{**} = T$

(*v*) $\|T\| = \|T^*\|$

(*vi*) $\|T^*T\| = \|T\|^2$.

Or

(*b*) Define invariant subspace.

(*i*) Show that, a closed linear subspace M of H is invariant under an operator T iff $M^\perp$ is invariant under T^*. 8

(*ii*) Show that, a closed linear subspace M of H reduces an operator T iff M is invariant under both T and T^*. 7

4. Attempt the following :

(*a*) (*i*) Let T be an arbitrary operator on H and $\lambda_1, \lambda_2, ...\lambda_n$ are the eigen values of T. $M_1, M_2, ... M_n$ are the corresponding eigen spaces and $P_1, P_2, ... P_n$ are the projections on these eigenspaces. If M_i's are pairwise orthogonal and M_i's spans H, then prove that, P_i's are pairwise orthogonal, $I = \sum_{i=1}^{n} P_i$ and $T = \sum_{i=1}^{n} \lambda_i P_i$. 8

(*ii*) If T is normal operator on H, then show that, M_i's spans H. 7

Or

(*b*) (*i*) If P_i's are pairwise orthogonal and $I = \sum_{i=1}^{n} P_i$ and $T = \sum_{i=1}^{n} \lambda_i P_i$, then prove that T is normal. 8

(*ii*) If T is normal operator on H, then show that, M_i's are pairwise orthogonal. 7

5. Attempt any *three* of the following : 5 marks each

(*a*) Show that, vector addition and scalar multiplication are jointly continuous

(*b*) Show that, the closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

(*c*) If T is an arbitrary operator on H and if α, β are scalars such that $|\alpha| = |\beta|$, then show that, $\alpha T + \beta T^*$ is normal operator on H.

(*d*) Define :

(*i*) Eigen value

(*ii*) Eigen vector

(*iii*) Eigen space.