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NEPNY—32—2023

FACULTY OF SCIENCE AND TECHNOLOGY

M.A./M.Sc. (NEP) (First Year) (First Semester) EXAMINATION

NOVEMBER/DECEMBER, 2023

MATHEMATICS

Paper–SMATC–402

(Real Analysis)

(Friday, 22-12-2023) **Time : 10.00 a.m. to 1.00 p.m.**

Time—3 *Hours* *Maximum Marks*—80

N.B. :— (*i*) *All* questions carry equal marks.

(*ii*) Q. No. **1** is compulsory.

(*iii*) Answer any *three* from Q. No. **2** to Q. Nos. **6**.

(*iv*) Figures to the right indicate full marks.

1. Answer the following : 20

(*a*) If $f \in R(\alpha)$ on $[a, b]$, then prove that $|f| \in R(\alpha)$ in $[a, b]$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f|\, d\alpha$.

P.T.O.

(*b*) Let $f_n(x) = \frac{\sin x}{\sqrt{n}}$, $(x \in R, n = 1, 2, ...)$, then prove that $\lim_{n\to\infty} f_n{}'(0) \neq f'(0)$.

(*c*) Every convergent sequence contains a uniformly convergent subsequence. Justify your answers.

(*d*) Let $f : R \to R$ with $f(0) =$ and $f(t) = t^2 \sin\left(\frac{1}{t}\right)$ if $t \neq 0$, then prove that f is differentiable on R but not of class c^1 on R.

2. Answer the following : 20

(*a*) Prove that, $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition p on $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

(*b*) (*i*) Suppose α increases on $[a, b]$, $a \le x_0 \le b$, α continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$, if $x \neq x_0$. Show that $f \in R(\alpha)$ and that $\int f \, d\alpha = 0$.

(*ii*) Does the integrability of $|f|$ implies that of f? Justify your answer.

3. Answer the following :

(*a*) State and prove the Weierstrass M-test for uniform convergence for series of function. 20

(*b*) (*i*) Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $(x \in R, n = 0, 1, 2, ...)$. Then prove that the series $\sum_{n=0}^{\infty} f_n(x)$ of continuous functions converges to a discontinuous sum.

(*ii*) Does every point convergence of sequence of functions is uniform convergent ? Justify your answer.

4. Answer the following : 20

(*a*) Given a double sequence $\{a_{ij}\}$. $i = 1, 2, 3$ ....., $j = 1, 2, 3$ .... , suppose that :

$$\sum_{j=1}^{\infty} |a_{ij}| = b_i \quad (i = 1, 2, 3 \ldots\ldots)$$

and $\sum b_i$ converges. Then prove that :

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

(*b*) (*i*) Does the every member of an equicontinuous family is uniformly continuous ? Justify your answer.

(*ii*) Suppose f is a real continuous function on R^1, $f_n(t) = f(nt)$ for $n = 0, 1, 2,$ .... and $\{f_n\}$ is equicontinuous on [0, 1]. What conclusions can we draw about f ?

5. Answer the following : 20

(*a*) (*i*) Let $A \subseteq R^m$, let $f : A \to R$, if f is differentiable at $\bar{a}$, then prove that $Df(\bar{a}) = [D_1 f(\bar{a}), D_2 f(\bar{a}), \ldots\ldots\ldots, D_m f(\bar{a})]$.

(*ii*) Let A be open in R^m, let $f : A \to R$, if f is differentiable on A. If A contains the line segment with end points $\bar{a}$ and $\bar{a} + \bar{h}$, then prove that there is a point $\bar{c} = \bar{a} + t_0 \bar{h}$ with $0 < t_0 < 1$, of this line segment such that $f(\bar{a} + \bar{h}) - f(\bar{a}), = Df(\bar{c}) . (\bar{h})$.

P.T.O.

(*b*) Given $f : R^5 \to R^2$ of class C^1. Let $\bar{a} = (1, 2, -1, +3, 0)$. Suppose that $f(\bar{a}) = \bar{0}$ and $Df(\bar{a}) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}$:

(I) Show there is a function $f : B \to R^2$ of class C^1 defined on an open set B of R^3 such that : $f(x_1, g_1(x), g_2(x), x_2 x_3) = \bar{0}$ for $\bar{x} = (x_1, x_2, x_3) \in B$ and $g(1, 3, 0) = (2, -1)$.

(II) Find $Dg(1, 3, 0)$.

6. Answer the following :

(*a*) If P^* is a refinement of P, then prove that : 20

$L(P, f, \alpha) \leq L(P^*, f, \alpha)$

and $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

Also, let $f_n(x) = n^2 x (1 - x^2)^n$, $0 \leq x \leq 1$ and $n = 1, 2, 3, \ldots\ldots$, show that :

$$\lim_{n\to\infty}\left[\int_0^1 f_n(x)dx\right] \neq \int_0^1\left[\lim_{x\to\infty} f_n(x)\right]dx$$

(*b*) State the Stone-Weierstrass theorem.

Let $f : R^2 \to R$ be defined by setting $f(0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2},$$

If $f(x, y) \neq 0$.

(*i*) For which vectors $u \neq 0$ does $f'(0, u)$ exist ? Evaluate it when it exists.

(*ii*) Do $D_1 f$ and $D_2 f$ exist at 0 ?

(*iii*) Is f differentiate at 0 ?

(*iv*) Is f continuous at 0 ?