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NY—133—2023

FACULTY OF ARTS/SCIENCE

M.A./M.Sc. (First Year) (Second Semester) EXAMINATION

NOVEMBER/DECEMBER, 2023

(New/CBCS)

MATHEMATICS

Paper-VIII

(Measure and Integration Theory)

(Friday, 8-12-2023) **Time : 10.00 a.m. to 1.00 p.m.**

Time—3 *Hours* *Maximum Marks*—75

Note :— (*i*) *All* questions are compulsory.

(*ii*) Figure to the right indicate full marks.

1. (*a*) Prove that, the outer measure of an interval equals its length. 15

Or

(*b*) State and prove Lebesgue's monotone convergence theorem. Also, show that for any set A, m^* (A) = m^* (A + x), where, A + x = [$y + x : y \in$ A]. 15

2. (*a*) Let $f \in$ BV [a, b], then prove that : $f(b) - f(a) = \text{P} - \text{N}$ and $\text{T} = \text{P} + \text{N}$, where all variations being in the finite interval [a, b].

Also, let f be defined by $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. $f(0) = 0$. Find the four derivative at $x = 0$. 15

P.T.O.

Or

(*b*) Let [*a*, *b*] be a finite interval and let $f \in L(a, b)$ with indefinite interval F, then prove that $F' = f$ a.e., in [*a*, *b*]. 15

3. (*a*) Let $\{A_i\}$ be a sequence in a ring R, then prove that there is a sequence $\{B_i\}$ of disjoint sets of R such that $B_i \subseteq A_i$ for each *i* and

$$\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$$

For each N, so that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Also, let $f = g$ *a.e.* (μ), where μ is a complete measure. Show that if f is measurable, so is g. 15

Or

(*b*) Let μ^* be an outer measure on **H (R)** and let S^* denote the class of μ^* measurable sets. Then prove that S^* is a σ-ring and μ^* restricted to S^* is a complete measure. 15

4. (*a*) State and prove the Jordan decomposition theorem. 15

Or

(*b*) Show that, a countable union of sets positive with respective to a signed measure v is a positive set. 15

Also, show that if

$$\phi(E)=\int_E f d\mu,$$

where $\int f d\mu$ is defined, then ϕ is a signed measure.

5. Attempt any *three* of the following : 5 each

(*a*) Show that, if m^* (A) = 0, then m^* (A $\cup$ B) = m^* (B), for any set B.

(*b*) Given an example where $D^+ (f + g) \phi D^+ f + D^+ g$.

(*c*) Define :

(*i*) Hereditary

(*ii*) Measure on a ring R.

(*d*) If $v(E)=\int_E f d\mu$ where $\int f d\mu$ exist, what is $|v|$ (E) ?